Euler Entertainments

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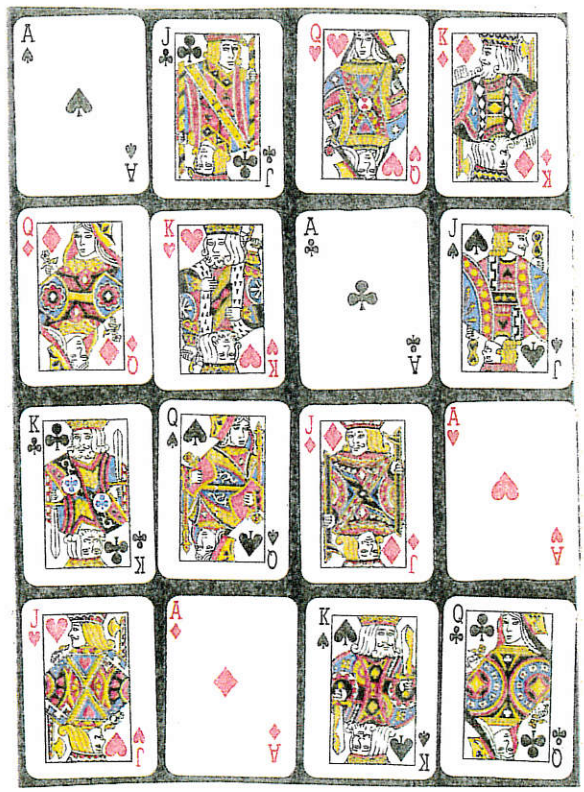
By Jeremiah and Karen Farrell

Martin Gardner's article "Euler's Spoilers: The Discovery of an Order-10 Graeco-Latin Square" Chapter 14 in his book *New Mathematical Diversions from Scientific American*, Simon & Schuster, 1966 reports on a certain puzzle traced back to 1624 in a book by Claude Gaspar Bachet:

> Arrange the sixteen highest playing cards so that no value or suit appears twice in any row, column or the two main diagonals. To Euler the values are Graeco and the suits Latin. Now days usually abbreviated to simply Latin squares.

One possible solution (of 144) follows.

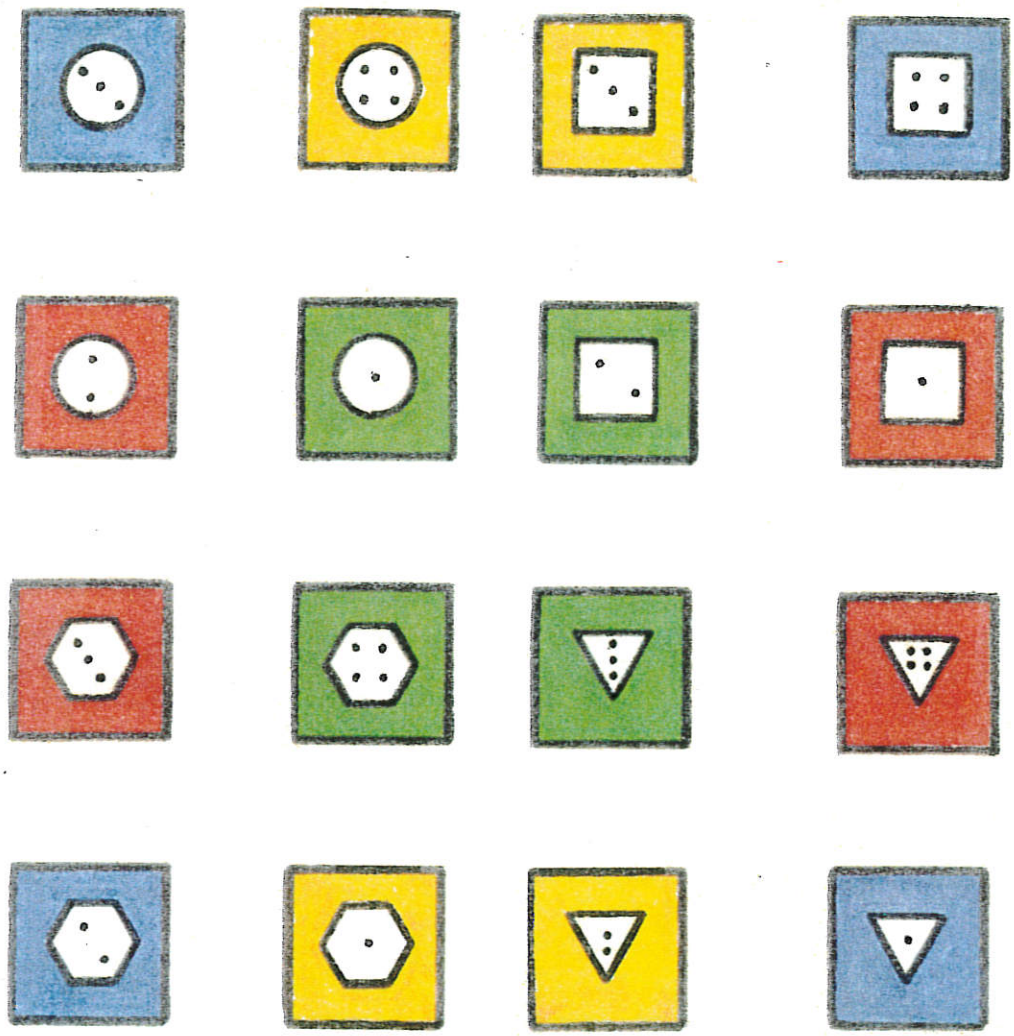


This square can be made magic with constant 34 by, say, labeling the values 1, 2, 3 and 4 and the suits 0, 4, 8 and 12 and then adding the pairs in each entry.

The two sets of four are what Leonard Euler (1707-1783) in the last years of his life called mutually orthogonal squares or 4x4 squares that could be superimposed so no two entries were duplicated. Euler tried to find such n x n Latin squares for all n. Euler knew that such squares exist if n is odd or if n = 4k but conjectured that no solution exists for n

= 4k+2. In 1901 Gaston Tarry published a proof no solution exists for n = 6, but E. T. Parker, R. C. Bose, and S. S. Shrikhand proved that Euler was wrong and n = 6 is the only exception. *Scientific American's* cover for November 1959 had staff artist Emi Kasdi's depiction of two order n = 10 Graeco-Latin squares.

Much more can be said about mutually orthogonal squares. Here is a list of sixteen entries combining three 4x4s using number, shape and color.



The reader is asked as a puzzle to arrange the pieces in a 4x4 square so that every row and column has no duplicated symbols. Euler called such squares semimagic and for order 4, three is the maximum number of mutually orthogonal squares possible.

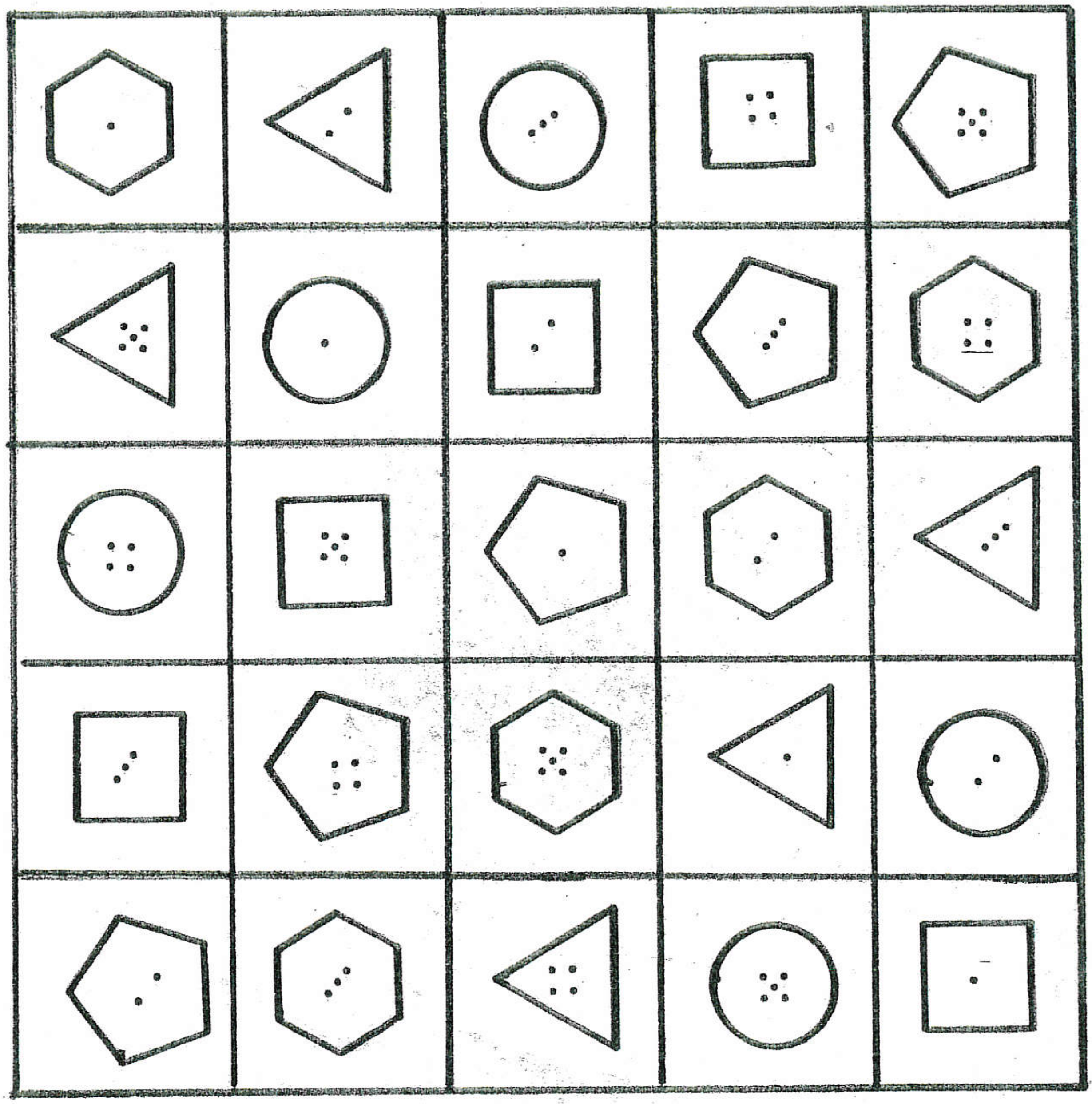
In fact for order n the maximum number of squares can only be n-1. This is not always obtained and to date it is not known for which orders this maximum is obtained.

For order 5 there can be the maximum 4 and it is possible to construct two puzzle-games using some of the possibilities. With the two such orthogonal pairs of number and shape shown, mix the 25 tokens face-down and each of two players draws 10 tokens. They will alternately place a token on a 5x5 board under one of the following two rules.
(1) No two tokens have a symbol in common in any row or column.
(2) (Cut-throat) Same as (1) but in addition no two tokens in any diagonal, broken or not, can have a symbol in common.

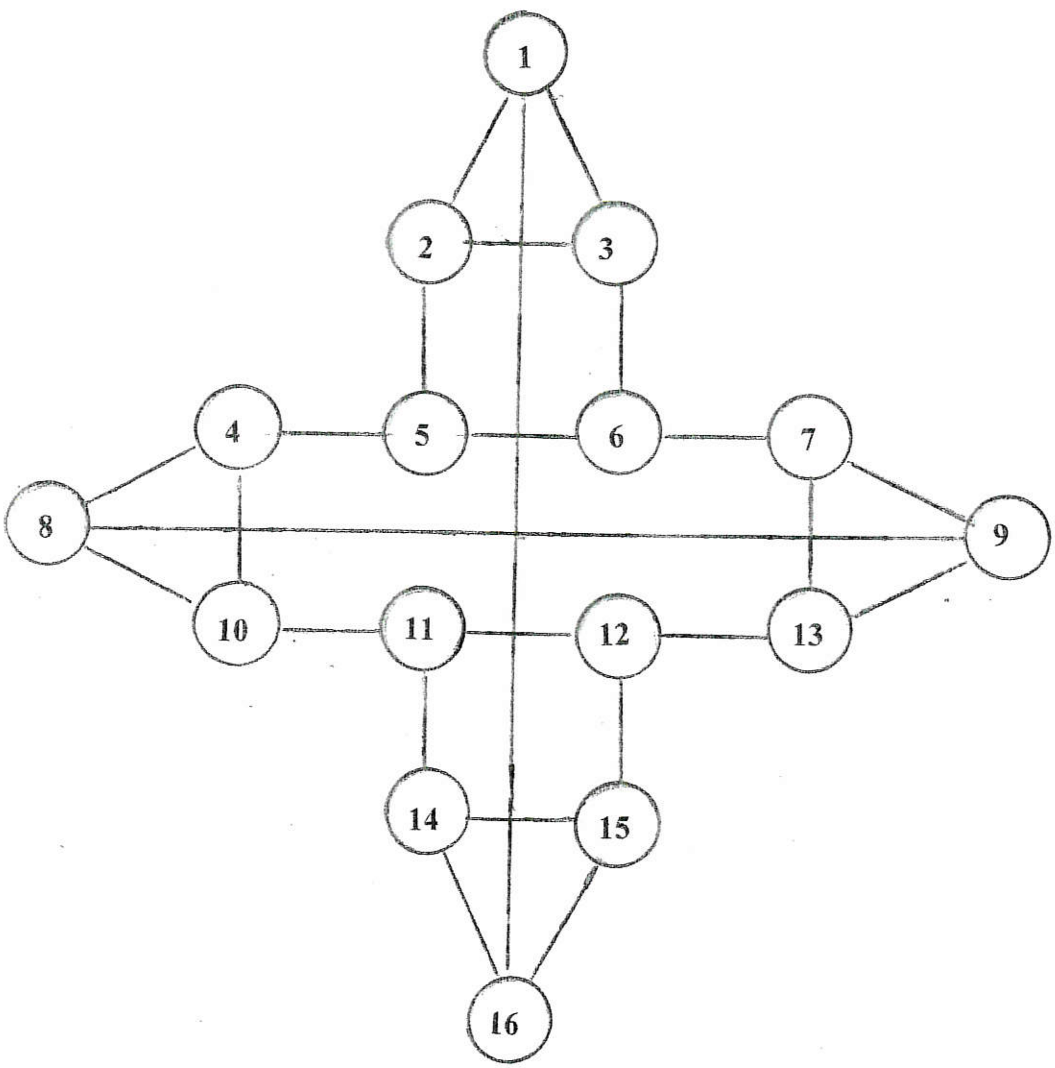
In either case, players can draw from the remaining pieces in the "kitty" if they cannot place one of their own. The onus is always on the second player to note misplays by the first player. The last player to be able to play wins.

There are 10 diagonals in any 5x5 and version (2) of the games will allow two more mutual orthogonals to reach the maximum four. Once again (2) will yield an ordinary magic square with constant 65 by labeling the shapes 0, 5, 10, 15, and 20 then adding the numbers to them in each entry.

Some solutions follow at the end of this article.



There are many more puzzles using the three order 4 tokens. For instance mathematicians call graphs in which each node has three edges on it Cubic Graphs. For 16 nodes 4207 non-isomorphic (essentially different) cubic graphs exist. Although the graphs are fundamentally different, the 16 tokens are rich enough to form somewhat challenging puzzles. In each of the following graphs place the tokens so that any connected tokens have a symbol in common. This is called a "hit" puzzle. It may also be possible in most cases to form a solution where connected tokens have no symbol in common, a "keepaway" puzzle.



THE CROSS

Cubic "Hit" Graph

Figure 1

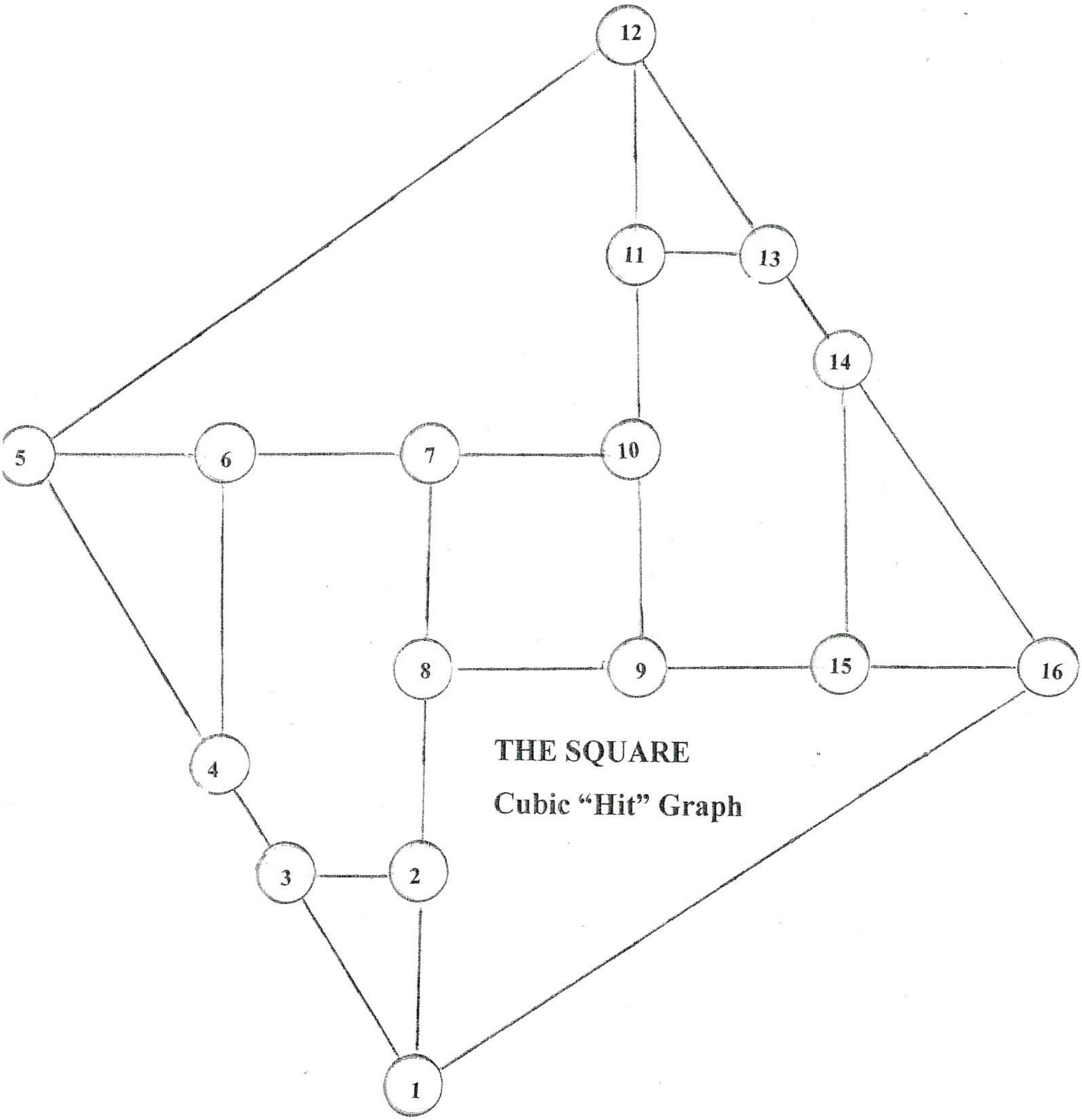


Figure 2

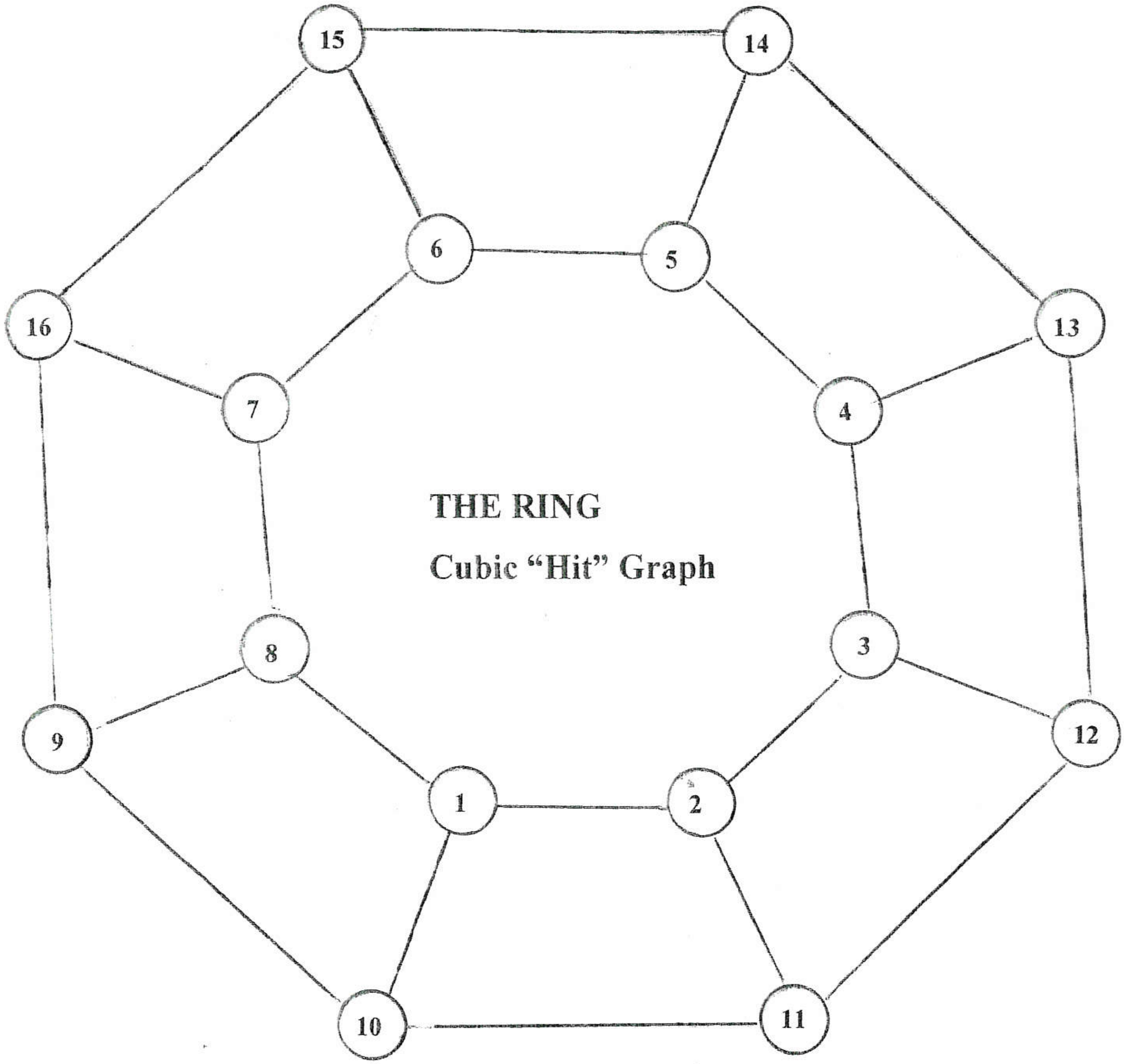
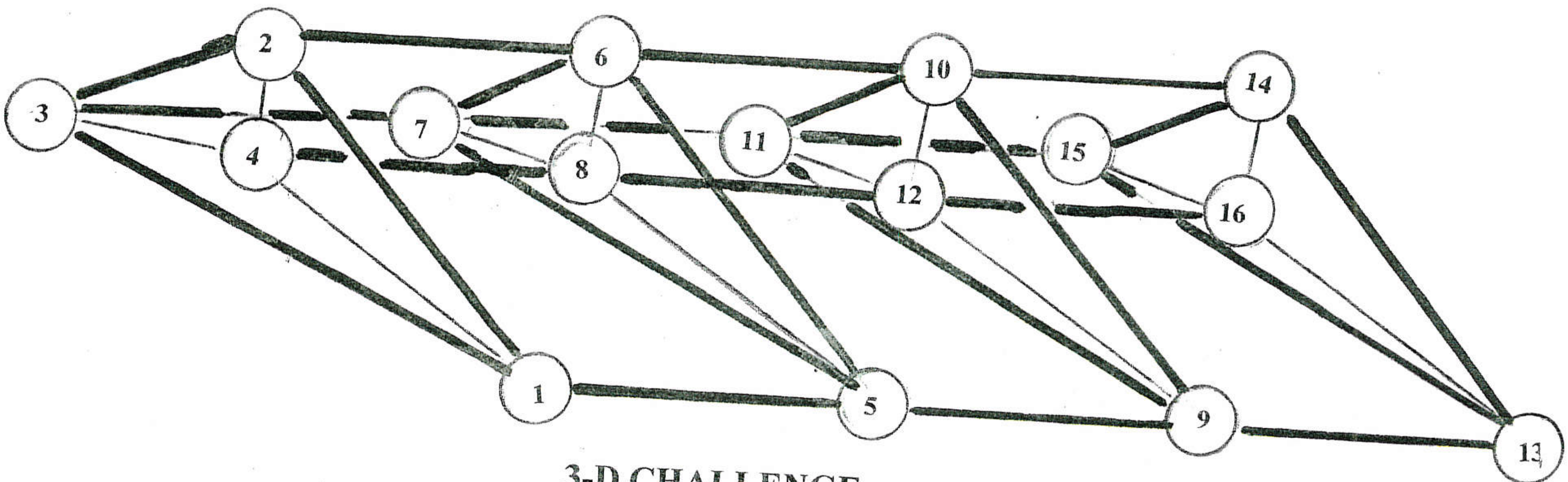


Figure 3



3-D CHALLENGE

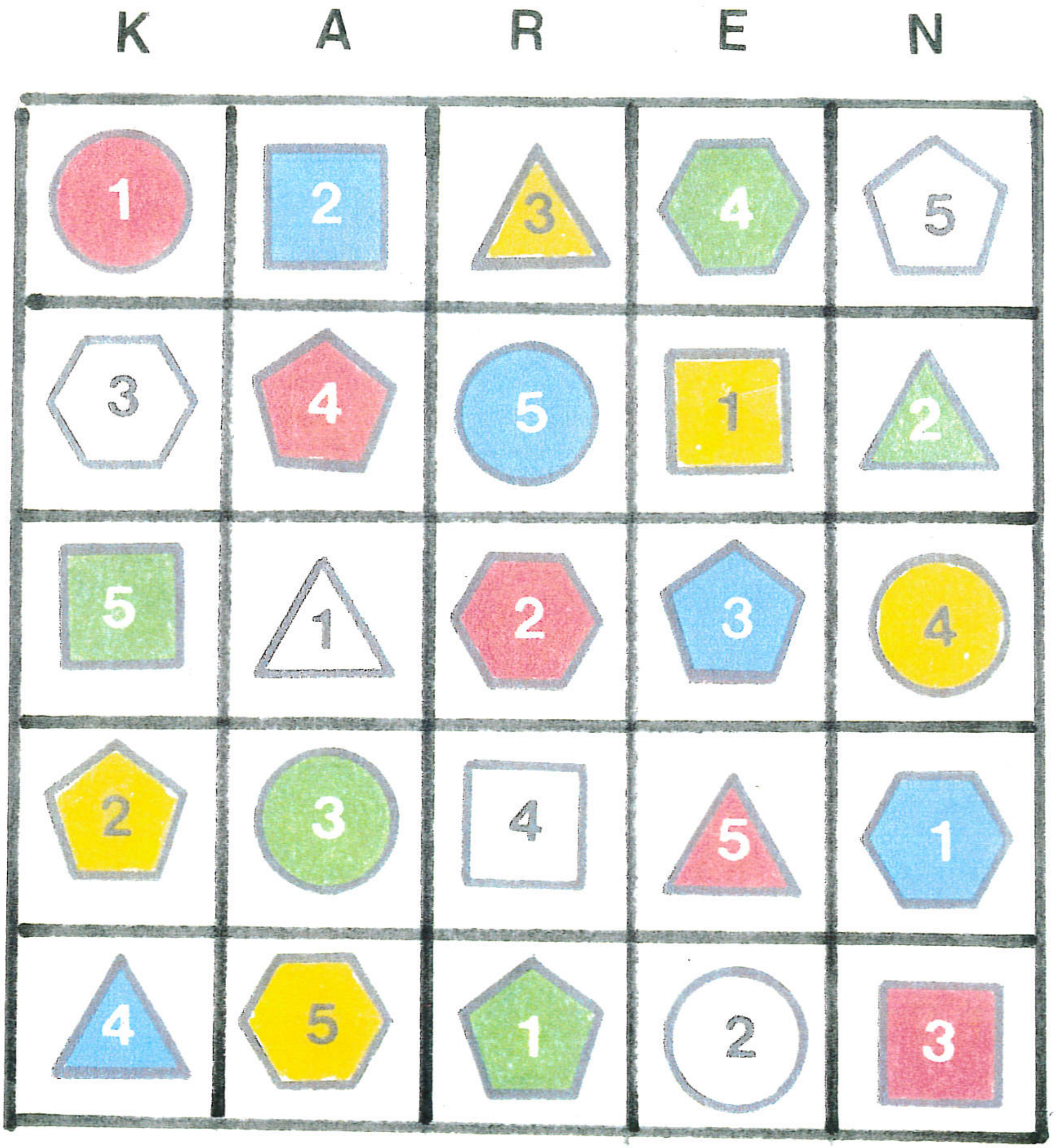
Play "Keep-Away" on

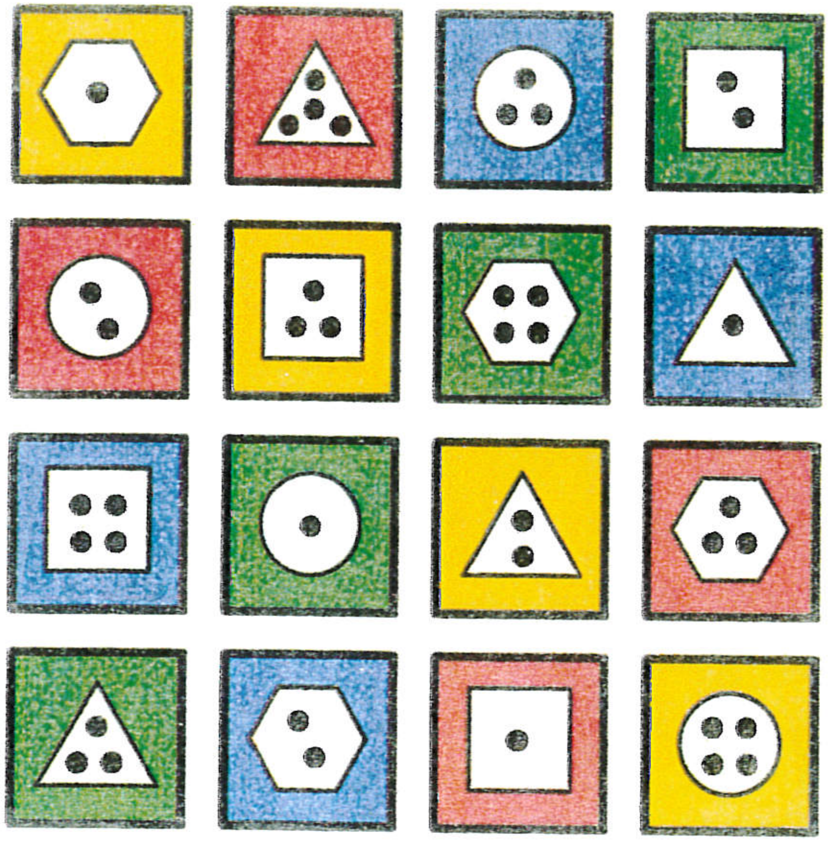
All Lines

Figure 4

ANSWERS

We define a knight sweep as chess knight moves from a corner of a square and the center of a 5x5. Trace for example in the playing card example how knight sweeps for the values and the suits evolve. In our other square examples likewise. For the 5x5 solution note the five colored diagonals. The five opposite diagonals can be labeled with the letters of KAREN.





Figures								
(1)	4	3	14	7	13	2	15	10
(2)	10	15	12	9	8	6	1	7
(3)	3	8	7	4	10	15	14	11
(4)	3	6	13	12	5	4	11	14

(1)	8	12	11	9	16	6	5	1
(2)	4	3	5	2	11	13	16	14
(3)	12	9	16	13	2	5	6	1
(4)	9	16	7	2	8	1	10	15

More cubic 16 puzzles or two person games. Either "HIT" or "KEEPAWAY"